Extremes on Phase-Type Distributions

Alexandru V. Asimit

Bruce L. Jones

University of Western Ontario

Summary

- Univariate results
- Bivariate results
- Examples
- Conclusions

Let $X_1, X_2, \ldots$ be iid r.vs with common df F.

Fisher-Tippett Theorem: Let $U^n = \max(X_i, i = 1, \ldots, n)$. If there exist a r.v U with nondegenerate df G and *normalizing constants* $a_n > 0$, b_n such that $a_n U^n + b_n \xrightarrow{w} U$, then G belongs to the type of one of the following

$$\begin{array}{lclll} \Phi_\alpha(x) & = & \exp\left(-x^{-\alpha}\right), & x > 0 & \Rightarrow G \text{ is of Fréchet type} \\ \Psi_\alpha(x) & = & \exp\left(-(-x)^{\alpha}\right), & x \leq 0 & \Rightarrow G \text{ is of Weibull type} \\ \Lambda(x) & = & \exp\left(-e^{-x}\right), & x \in \Re & \Rightarrow G \text{ is of Gumbel type} \end{array}$$

where $\alpha > 0$, and we write $F \in MaxDA(G)$.

Introduction: Univariate Minima

Let $L^n = \min(X_i, i = 1, \ldots, n)$. If there exist a r.v L with nondegenerate df G and normalizing constants $a_n > 0$, b_n such that $a_n L^n + b_n \xrightarrow{w} L$, then G belongs to the type of one of the following

$$
\begin{aligned}
\Phi^*_\alpha(x) &= 1 - \exp\left(-(-x)^{-\alpha}\right), & x < 0 &\Rightarrow G \text{ is of type I} \\
\Psi^*_\alpha(x) &= 1 - \exp\left(-x^{\alpha}\right), & x \geq 0 &\Rightarrow G \text{ is of type II} \\
\Lambda^*(x) &= 1 - \exp\left(-e^{x}\right), & x \in \Re &\Rightarrow G \text{ is of type III}
\end{aligned}
$$

where $\alpha > 0$, and we write $F \in MinDA(G)$.

Univariate Phase-Type Distribution

Let $\{Y(t),\, t \geq 0\}$ be a CTMC with state space $\xi = \{\Delta, 1, \ldots, d\}$, initial distribution $\boldsymbol{\beta} = (0, \boldsymbol{\alpha})$, and infinitesimal generator

$$\mathbf{Q} = \begin{pmatrix} 0 & \mathbf{0} \\ \textbf{-Ae} & \mathbf{A} \end{pmatrix}$$

Then the nonnegative random variable X of the time until absorption into state Δ is $PH(\boldsymbol{\alpha}, \mathbf{A}, d)$.

$$\bar{F}(x) = \Pr(Y(x) \notin \{\Delta\}) = \boldsymbol{\alpha} e^{\mathbf{A}x}\mathbf{e}, \quad x \geq 0.$$

The matrix **A** has a real dominant eigenvalue $-\eta$, not necessarily unique, such that for all complex eigenvalues λ, $\mathrm{Re}(\lambda) < -\eta$.

1. if $-\eta$ is a simple eigenvalue of **A** then

$$e^{\mathbf{A}x} = e^{-\eta x}(\mathbf{M} + \boldsymbol{O}(1)),\ as\ x \to \infty$$

2. if $-\eta$ has algebric multiplicity l, then there exists $k \in [0, l-1]$

$$e^{\mathbf{A}x} = x^k e^{-\eta x}(\mathbf{M} + \boldsymbol{O}(1)),\ as\ x \to \infty,$$

where $k+1$ is the maximal order of Jordan blocks corresponding to $-\eta$, called the index of $-\eta$.

Let X be a PH$(\boldsymbol{\alpha}, \mathbf{A}, d)$ random variable. Then $F \in MaxDA(\Lambda)$ with normalizing constants

$$a_n = \frac{1}{\eta}, b_n = \frac{\log nc + k \log \log n - k \log \eta}{\eta},\ \text{where } c = \boldsymbol{\alpha}\mathbf{Me} > 0.$$

Minima: Univariate Case

Let X is PH$(\boldsymbol{\alpha}, \mathbf{A}, d)$. Then m is the minimum number of transitions needed for the underlying CTMC to be absorbed if and only if

$$-\boldsymbol{\alpha}\mathbf{A}^m\mathbf{e} > 0 \text{ and when } m \geq 2, \; -\boldsymbol{\alpha}\mathbf{A}^\ell\mathbf{e} = 0, \; \ell = 1, \ldots m-1.$$

Let X be a PH$(\boldsymbol{\alpha}, \mathbf{A}, d)$. Then $F \in MinDA(\Psi_m^*)$ with normalizing constants

$$a_n = \left(\frac{m!}{nc}\right)^{1/m}, \quad b_n = 0,$$

where $c = -\boldsymbol{\alpha}\mathbf{A}^m\mathbf{e}$.

- $\{Y(t),\ t \geq 0\}$ is a CTMC with finite state space $\xi = \{\Delta, 1, \ldots, d\}$, initial distribution $\boldsymbol{\beta} = (0, \boldsymbol{\alpha})$ and infinitesimal generator **Q**

- $\xi_i,\ i = 1, \ldots, p$, are nonempty stochastically closed subsets of the state space ξ such that $\bigcap_{i=1}^{p} \xi_i = \{\Delta\}$, and

$$\xi = \left(\bigcup_{i=1}^{p} \xi_i\right) \bigcup \xi_0 \text{ for some subset } \xi_0 \subset \xi \text{ with } \xi_0 \bigcap \xi_i = \emptyset, i = 1, \ldots, p$$

- $X_i = \inf\{t \geq 0 : Y(t) \in \xi_i\},\ i = 1, \ldots, p$

The joint distribution of $(X_1, \ldots, X_p)$ is called a MPH random vector with representation $(\boldsymbol{\alpha}, \mathbf{A}, \xi, \xi_1, \ldots, \xi_p)$. Thus, a MPH distribution is a joint distribution of first passage times to various overlapping subsets of the state space ξ.

Multivariate Phase-Type Distribution (cont'd)

For $0 \le x_p \le \cdots \le x_1$

$$\bar{F}(x_1, \ldots, x_p) = \boldsymbol{\alpha}\, e^{\mathbf{A}x_p}\, \mathbf{g}_p\, e^{\mathbf{A}(x_{p-1}-x_p)}\, \mathbf{g}_{p-1} \cdots e^{\mathbf{A}(x_1-x_2)}\, \mathbf{g}_1 \mathbf{e},$$

where, for $k = 1, \ldots, p$, $\mathbf{g}_k$ is a $d \times d$ diagonal matrix whose ith diagonal entry, for $i = 1, \ldots, d$, equals 1 if $i \in \xi \setminus \xi_k$ and 0 otherwise.

The random variable X_i represents the first passage time of the CTMC into ξ_i. This implies that X_i is univariate PH distributed with representation $(\boldsymbol{\alpha}_{\xi\setminus\xi_i}, \mathbf{A}_{\xi\setminus\xi_i}, d + 1 - |\xi_i|)$

Bivariate Phase-Type Distribution

$$\mathbf{A} = \begin{pmatrix} \mathbf{A_0} & \mathbf{B_1} & \mathbf{B_2} \\ 0 & \mathbf{A_1} & 0 \\ 0 & 0 & \mathbf{A_2} \end{pmatrix},$$

where, $\mathbf{A}_i$ represents the subgenerator for states in $\xi_i \setminus \{\Delta\}$, and $\mathbf{B}_i$ represents the matrix of transition intensities from states in ξ_0 to states in $\xi_i \setminus \{\Delta\}$.

Example: Marshall-Olkin df has subgenerator

$$\mathbf{A} = \begin{pmatrix} -\lambda_{12} - \lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 \\ 0 & -\lambda_{12} - \lambda_2 & 0 \\ 0 & 0 & -\lambda_{12} - \lambda_1 \end{pmatrix}$$

$\bar{F}(x, y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)\}$

Multivariate Maxima/Minima

Let $\mathbf{X}^{(1)} = (X_1^{(1)}, \ldots, X_p^{(1)})$, $\mathbf{X}^{(2)} = (X_1^{(2)}, \ldots, X_p^{(2)}), \ldots$ be iid random vectors with common distribution F, and let $\mathbf{U}^{(n)}$ be a random vector with jth component

$$U_j^{(n)} = \max(X_j^{(i)}, i = 1, \ldots n).$$

If there exist $\mathbf{a}^{(n)}$, $\mathbf{b}^{(n)} \in \Re^p$ and $\mathbf{U}$ with df G such that

$$\mathbf{a}^{(n)}\mathbf{U}^{(n)} + \mathbf{b}^{(n)} \xrightarrow{w} \mathbf{U},$$

then $F \in \text{MaxDA}(G)$.

Theorem 1 *Let F be the distribution function of a bivariate PH distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{A}, \xi, \xi_1, \xi_2)$, and m_i be the minimum number of transitions required in order to enter ξ_i. Then the limiting distribution of the componentwise minima is given by*

- *Case 1:* $m_1 = m_2 = m$

$$\bar{G}(x_1, x_2) = \exp\left\{-x_1^m - x_2^m + c\min\left(\frac{x_1^m}{c_1}, \frac{x_2^m}{c_2}\right)\right\},$$

where $c_i = -\boldsymbol{\alpha}\boldsymbol{A}^{m_i}\boldsymbol{g}_i\boldsymbol{e}$, $i = 1, 2$, *and* $c = -\boldsymbol{\alpha}\boldsymbol{A}^m\boldsymbol{e}$.

- *Case 2:* $m_1 \neq m_2$

$$\bar{G}(x_1, x_2) = \exp(-x^{m_1} - x^{m_2})$$

Theorem 2 *Let F be the distribution function of a bivariate PH distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{A}, \xi, \xi_1, \xi_2)$. Then the limiting distribution of the componentwise maxima has the following form:*

$$G(x_1, x_2) = \begin{cases} e^{-e^{-x_1}} e^{-e^{-x_2}} \exp\left\{ \frac{e^{-x_1}}{c_1} \boldsymbol{\alpha} \boldsymbol{M}_1 e^{\boldsymbol{A}\left(x_2 + \log c_2 - x_1 - \log c_1\right)\eta^{-1}} \boldsymbol{g}_2 \boldsymbol{e} \right\}, \\ \qquad \text{if } x_1 + \log c_1 \le x_2 + \log c_2 \\ e^{-e^{-x_1}} e^{-e^{-x_2}} \exp\left\{ \frac{e^{-x_2}}{c_2} \boldsymbol{\alpha} \boldsymbol{M}_2 e^{\boldsymbol{A}\left(x_1 + \log c_1 - x_2 - \log c_2\right)\eta^{-1}} \boldsymbol{g}_1 \boldsymbol{e} \right\}, \\ \qquad \text{if } x_2 + \log c_2 \le x_1 + \log c_1 \end{cases}$$

if $\eta_1 = \eta_2 = \eta$ and $k_1 = k_2 = k$, where $c_i = \boldsymbol{\alpha} \boldsymbol{M}_i \boldsymbol{e} > 0$ for $i = 1, 2$.

For any other case we have independence, and

$$G(x_1, x_2) = \exp\left(-e^{-x_1}\right) \exp\left(-e^{-x_2}\right).$$

Pickands' representation

In the bivariate case, if $F \in \text{MaxDA}(G)$

$$G(x, y) = \exp\left\{\log(G_1(x)G_2(y))A\left(\frac{\log G_1(x)}{\log(G_1(x)G_2(y))}\right)\right\},$$

where A is the Pickands' representation function, which is a convex function on $[0, 1]$ such that $\max(t, 1-t) \leq A(t) \leq 1$.

$$A(t) = \begin{cases} 1 - \frac{1-t}{c_2}\boldsymbol{\alpha}\mathbf{M}_2 e^{\mathbf{A}\frac{1}{\eta}\log\frac{c_1}{c_2}\frac{1-t}{t}}\mathbf{g}_1\mathbf{e}, & \text{if } 0 \leq t \leq \frac{c_1}{c_1+c_2} \\ 1 - \frac{t}{c_1}\boldsymbol{\alpha}\mathbf{M}_1 e^{\mathbf{A}\frac{1}{\eta}\log\frac{c_2}{c_1}\frac{t}{1-t}}\mathbf{g}_1\mathbf{e}, & \text{if } \frac{c_1}{c_1+c_2} \leq t \leq 1 \end{cases}.$$

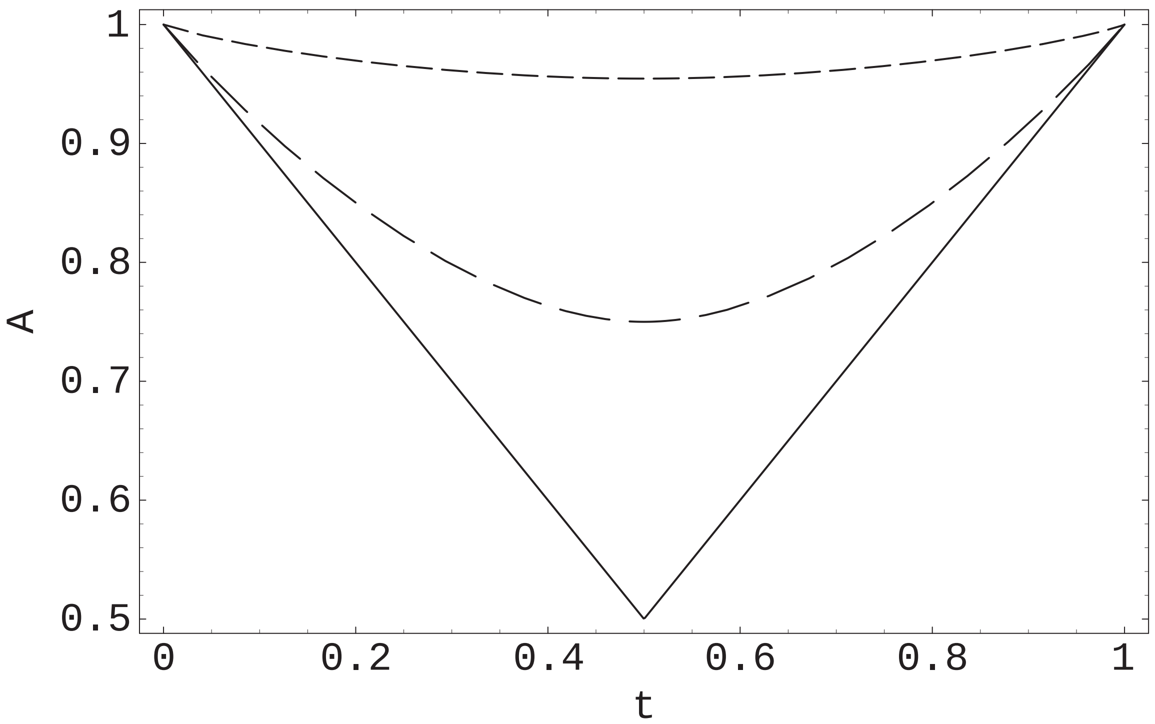
Example 1

$$\boldsymbol{\alpha} = (1,0,0), \quad \mathbf{A} = \begin{pmatrix} -a & p & q \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix}, \quad a < \min(b,c), p+q \leq a,$$

$$A(t) = \begin{cases} 1 - t + \left(\frac{b-a}{(c-a)(b+p-a)}\right)^{1-\frac{c}{a}} q(c+q-a)^{-\frac{c}{a}} t^{\frac{c}{a}} (1-t)^{1-\frac{c}{a}}, & 0 \leq t \leq \frac{c_1}{c_1+c_2} \\ t + \left(\frac{c-a}{(b-a)(q+c-a)}\right)^{1-\frac{b}{a}} p(p+b-a)^{-\frac{b}{a}} t^{1-\frac{b}{a}} (1-t)^{\frac{b}{a}}, & \frac{c_1}{c_1+c_2} \leq t \leq 1 \end{cases}.$$

If $p = q = 0$, then $A(t) = \max(t, 1-t)$

Example 1 (a)

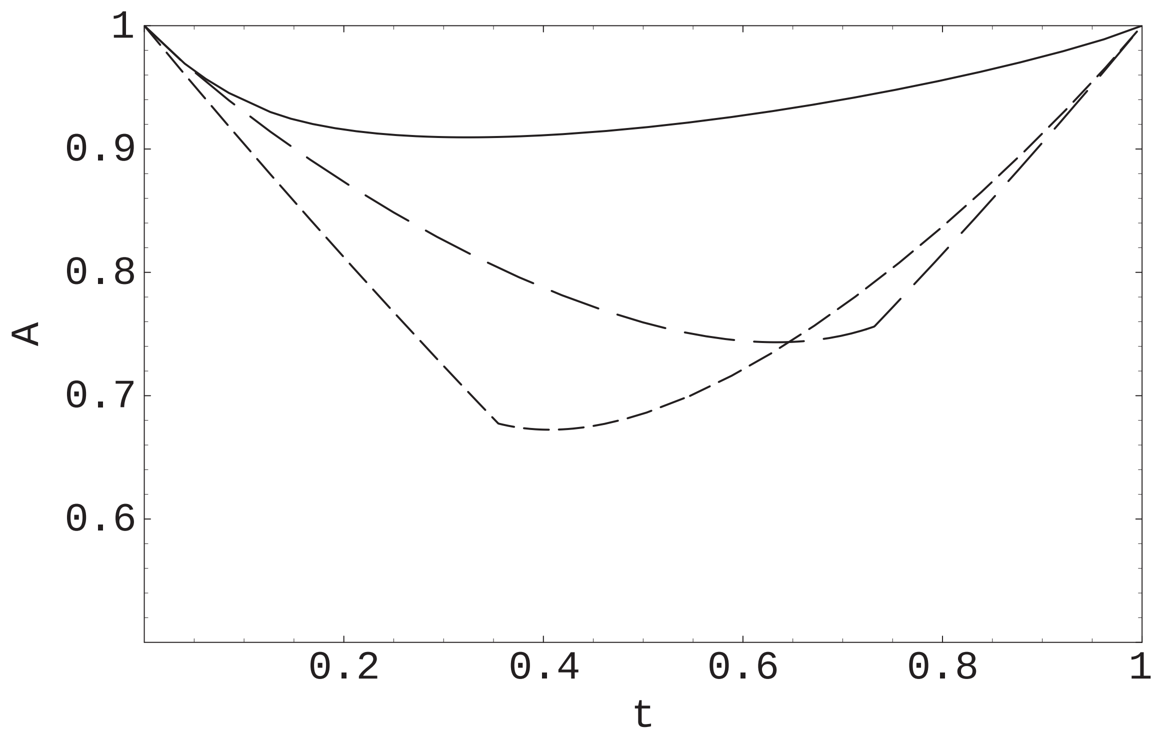


$(a, b, c, p, q) = (2, 3, 3, 0, 0) \longrightarrow$ solid line

$(a, b, c, p, q) = (2, 3, 3, 1, 1) \longrightarrow$ long-dashed line

$(a, b, c, p, q) = (2, 2.1, 2.1, 1, 1) \longrightarrow$ short-dashed line

Example 1 (b)



$(a, b, c, p, q) = (2, 2.1, 3, 1, 1) \rightarrow$ solid line

$(a, b, c, p, q) = (2, 3, 2.5, 0.1, 1) \rightarrow$ long-dashed line

$(a, b, c, p, q) = (2, 3, 3, 1, 0.1) \rightarrow$ short-dashed line

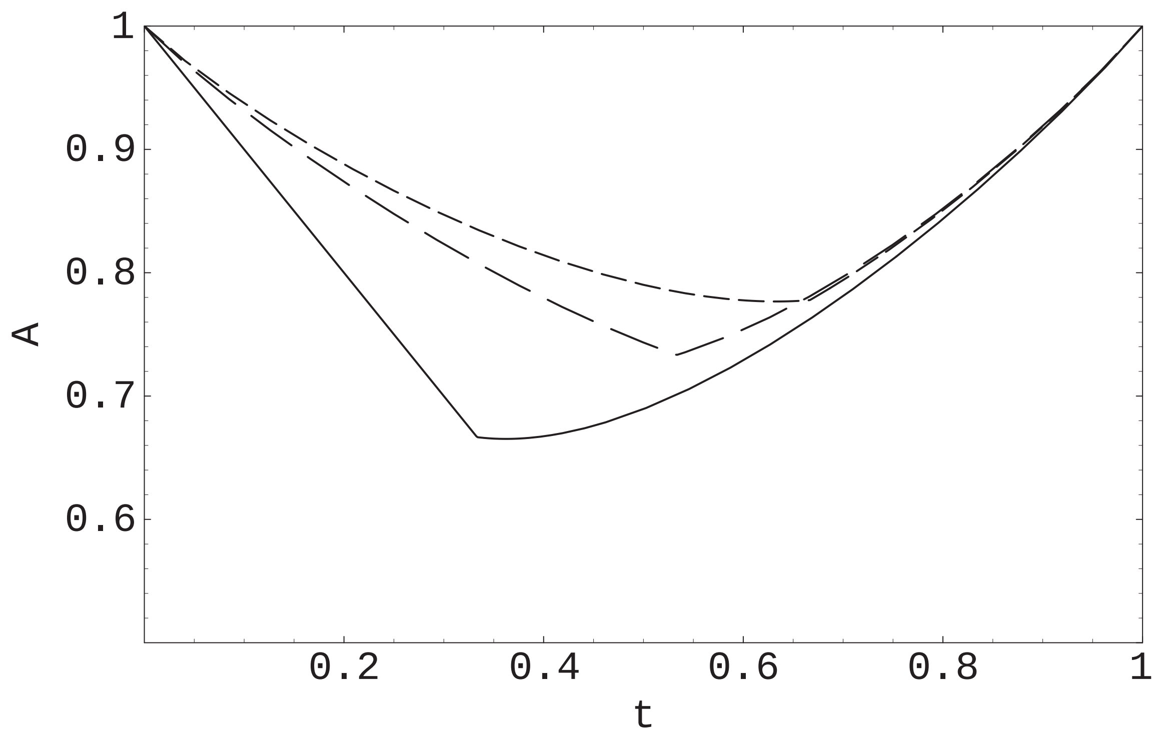
Example 2

$$\boldsymbol{\alpha} = (p, 1-p, 0, 0),\ 0 \le p \le 1,$$

$$\mathbf{A} = \begin{pmatrix} -5 & 0 & 1 & 2 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix},$$

$$A(t) = \begin{cases} 1 - t + 2^{\frac{4}{5}} p (1+2p)^{-\frac{6}{5}} (4-p)^{\frac{1}{5}} t^{\frac{6}{5}} (1-t)^{-\frac{1}{5}}, & 0 \le t \le \frac{2+4p}{6+3p} \\ t + 2^{\frac{2}{5}} (2-p)(4-p)^{-\frac{7}{5}} (1+2p)^{\frac{2}{5}} t^{-\frac{2}{5}} (1-t)^{\frac{7}{5}}, & \frac{2+4p}{6+3p} \le t \le 1 \end{cases}.$$

Example 2 (cont'd)



$p = 0 \longrightarrow$ solid line

$p = 0.5 \longrightarrow$ long-dashed line

$p = 1 \longrightarrow$ short-dashed line

Conclusions

- for bivariate maxima, a flexible Pickands' representation is obtained
- for minima, the bivariate exponential Marshall-Olkin df arises in the limit
- simpler to see in the bivariate case, but our results can be extended to higher dimensions